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Noncommutative Geometry, Strings and Duality

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Abstract

In this talk, based on work done in collaboration with G. Landi and R.J Szabo, I will review how string theory can be considered as a noncommutative geometry based on an algebra of vertex operators. The spectral triple of strings is introduced, and some of the string symmetries, notably target space duality, are discussed in this framework.

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It is a common belief that at distances of the order of Planck's length, where neither quantum mechanics nor general relativity can be considered perturbations of classical physics, a change in the very structure of spacetime will be necessary, and *classical geometry* will no longer be the appropriate tool. A suitable candidate for the mathematics: is *Noncommutative Geometry*, [1], a theory which substitutes the study of *classical* notions such as point, line etc. with the study of the algebras defined on Hausdorff topological spaces, with the obvious generalization given by noncommutative algebras. I will not introduce Noncommutative Geometry here, there are excellent reviews, and it is covered in several of the contributions to this Arbeitstagung.

On the physics side, a candidate to describe the theory of physical interactions at the Planck length is *String Theory* [2]. An introduction of strings will also be a task too formidable for this short talk, but the basic motivation behind string theory as a fundamental theory of spacetime, historically has been the presence in the spectrum of the theory of a spin two massless particle, identified with the graviton. String theory is a two-dimensional theory in which spacetime coordinates are fields on a 2-d surface, the world-sheet of the string. Interactions, that is joining and splitting of strings, is described by higher genus surfaces. At very high energies (of the order of Planck's energy), "strange things" begin to happen *∞ -genus surfaces, duality, branes whose coordinates are (noncommuting) matrices...*

In this talk I will try to construct the *Noncommutative Geometry of String Theory*. The key idea is due to Fröhlich and Gawędzki [3], and in this talk I will mainly sketch the developments of [4]-[10], to which I refer for further details and references.

String theory is described by a conformally invariant field theory on a 2-dimensional surface, the world-sheet, which is to be interpreted as the analog of the world line, swept by the string in its motion. Spacetime coordinates appear as *fields* on this two dimensional surface, which is usually assumed to be compact. We will consider bosonic strings, compactified on a d dimensional torus, \mathbb{R}^d quotiented by an abelian infinite group (a lattice) Γ generated by d generators e_i . On the generators of Γ we define an inner product which provides a metric (of Euclidean signature) on the torus \mathcal{T}_d :

$$\langle e_i, e_j \rangle \equiv g_{ij} \quad . \quad (1)$$

The dual lattice $\tilde{\Gamma}$ is spanned by the basis e^i with (we implicitly complexify Γ and extend the product):

$$\langle e^i, e_j \rangle = \delta_j^i \quad . \quad (2)$$

The inner products of the e^i 's define a metric which is the inverse of g_{ij} , that is:

$$\langle e^i, e^j \rangle \equiv g^{ij} \quad . \quad (3)$$

Notice that, if all of the e_i are quantities of order R (we take Planck's length to be unit unless otherwise stated), with $\det g$ of order R^d , then the 'size' of the dual lattice is a quantity of order $1/R$. In this sense, if to a given lattice corresponds a large universe, to its dual it will correspond a small one, the dual torus \tilde{T}_d^\dagger .

Classically the string is described by a two dimensional nonlinear σ model, whose fundamental objects are the Fubini–Veneziano fields, which, for the case of a closed string are:

$$X^i(\tau, \sigma) = x^i + g^{ij} p_i \tau + g^{ij} w_j \sigma + \sum_{k \neq 0} \frac{1}{ik} \alpha_k^{(\pm)i} e^{ik(\tau \pm \sigma)} \quad , \quad (4)$$

where x represents the centre of mass of the string, p its momentum and w is the winding number, the number of times the string wraps around the direction defined by the e_i . Notice that, since the space is compact, the momentum is quantized, and in fact it must be $p \in \tilde{\Gamma}$, while the winding number must belong to the dual lattice $w \in \Gamma$. If the size of the target space is extremely large, then the momentum will have a spectrum with very close eigenvalues, a nearly continuous spectrum, while the windings will have values far apart. But apart from these scale considerations, the role of p and w in (4) is symmetric. In the following we will concentrate on the zero modes of the string, mostly ignoring the oscillator modes. These are internal excitations of the other string, and are not sensible to the target space in which the strings live, and will therefore play an indirect role for the structure of spacetime. Moreover, the oscillators describe excitations which are starting at the Planck mass, while most of our considerations relate to the low energy sector of the theory. however they cannot be ignore altogether, as they reveal the “string” character of the theory.

[†]Strictly speaking however this conclusion is only valid only in the absence of torsion (introduced below) in the action [11].

We have therefore a nonlinear σ model described by the action:

$$S = \frac{1}{4\pi} \int d\sigma d\tau \sqrt{\eta} \eta^{\alpha\beta} \partial_\alpha X^i g_{ij} \partial_\beta x^j + \varepsilon^{\alpha\beta} b_{ij} \partial_\alpha X^i \partial_\beta X^j \quad , \quad (5)$$

where η is the world sheet two dimensional metric, ε is the antisymmetric tensor with $\varepsilon_{12} = 1$, G is the metric defined in (1), and b is an antisymmetric tensor which represent the ‘torsion’ of the string.

We can perform a chiral decompositions of the X ’s defining:

$$X_\pm^i(\tau \pm \sigma) = x_\pm^i + g^{ij} p_j^\pm(\tau \pm \sigma) + \sum_{k \neq 0} \frac{1}{ik} \alpha_k^{(\pm)i} e^{ik(\tau \pm \sigma)} \quad . \quad (6)$$

The zero modes x_\pm^i (the centre of mass coordinates of the string) and the (centre of mass) momenta $p_i^\pm = 2\pi p_i \pm (g - \mp b)_{ij} w^j$ are canonically conjugate variables,

$$[x_\pm^i, p_i^\pm] = -i\delta_i^j \quad (7)$$

with all other commutators vanishing. The left-right momenta are

$$p_i^\pm = \frac{1}{\sqrt{2}} (p_i \pm \langle e_i, w \rangle) \quad (8)$$

The p^\pm ’s belong to the lattice:

$$\Lambda = \tilde{\Gamma} \oplus \Gamma \quad (9)$$

We can therefore define the fields $X = X_+ + X_-$, and we may equally well define $\tilde{X} \equiv X_+ - X_-$, whose zero mode we will indicate as \tilde{x} .

Exchange of a lattice with its dual is a symmetry called T-duality [12, 13]. It corresponds to an exchange of the momentum quantum number with the winding, and of the zero mode corresponding to x , the position of the centre of mass of the string with its dual \tilde{x} . This is a symmetry of the Hamiltonian:

$$\begin{aligned} H &= \frac{1}{2} ((2\pi)^2 p_i g^{ij} p_j + w^i (g - b g^{-1} b)_{ij} w^j + 4\pi w^i b_{ik} g^{kj} p_j) \\ &\quad + \sum_{k>0} g_{ij} \alpha_{-k}^{(+i)} \alpha_k^{(+j)} + \sum_{k>0} g_{ij} \alpha_{-k}^{(-i)} \alpha_k^{(-j)} - \frac{d}{12} \end{aligned} \quad (10)$$

$$= \frac{1}{2} (p_+^2 + p_-^2) + \text{Oscillators} - \frac{d}{12} \quad (11)$$

with the term $-\frac{d}{12}$ due to normal ordering on quantization. Since momenta and windings belong to a lattice the spectrum is discrete.

Two target spaces related by a T-dual transformation are indistinguishable at low energies. This can be seen heuristically as follows [14]: in ordinary quantum mechanics position is just a derived concept, as the Fourier transform of momentum spaces. In a string theory is possible however to consider winding (and its eigenstates) rather than momentum. If the compactification radius is of the order one, the two choices are equivalent, but for a very large radius the eigenvalues of momentum are nearly continuous, while the ones of winding are far apart, the first one above zero being at a very large energy. It is therefore difficult to make “localized wave packets” with the Fourier transform of winding. Conversely, with a small radius of compactification, it is the winding which gives the possibility to create localized wave packets.

In the torsionless case, $b = 0$, this corresponds to an exchange of g with its inverse g^{-1} , and the change of size of the target space in which the radius $R \rightarrow 1/R$. In the presence of torsion the exchange is $g^{-1} \leftrightarrow g - bg^{-1}b$ and $bg^{-1} \leftrightarrow -g^{-1}b$, and it depends crucially on the values of the b_{ij} . In the toroidal case it is possible to exchange only some of the generators of the lattice with their duals, giving rise to a group of factorized T-dualities.

The full group of symmetry is even larger: it is in fact $O(d, d, \mathbb{Z})$ [15, 13], generated from three kinds of transformations:

- The factorized dualities we have already discussed.
- The changes of base of the lattices, made via a matrix which belongs to $G(d, \mathbb{Z})$, the group of integer valued matrices of unit determinant.
- The transformation $b_{ij} \rightarrow b_{ij} + c_{ij}$ with c an antisymmetric tensor with integer entries.

There is a further \mathbb{Z}_2 symmetry obtained exchanging σ and τ on the world sheet, this last symmetry does not affect the target space.

The theory has also two continuous symmetries:

- Target Space reparametrization:

$$X_{\pm}(z_{\pm}) \rightarrow X_{\pm}(z_{\pm}) + \delta X_{\pm}(z_{\pm}) \quad (12)$$

- World sheet conformal invariance, represented by two Virasoro algebras:

$$\begin{aligned} [L_k^\pm, L_m^\pm] &= (k-m)L_{k+m}^\pm + \frac{c}{12}(k^3-k)\delta_{k+m,0} \\ [L_k^-, L_m^+] &= 0, \end{aligned} \quad (13)$$

both these symmetries play a crucial role in the theory.

Given this scenario we want to construct the *Noncommutative Geometry* of interacting strings. We will therefore construct, in the spirit of Connes, a spectral triple, the Fröhlich-Gawędzki Spectral Triple [3]

Let us first construct the Hilbert Space of String states. Upon first quantization the oscillator modes become creation and annihilation operators:

$$[\alpha_k^{(\pm)i}, \alpha_m^{(\pm)j}] = kg^{ij}\delta_{k+m,0} \quad (14)$$

while the zero modes have the usual commutation relations (7). The Hilbert Space of (excited) string states therefore is:

$$\mathcal{H} = L_2(sp(T^d))^\Gamma \otimes \mathcal{F}^+ \otimes \mathcal{F}^- \quad (15)$$

where $L_2(sp(T^d))^\Gamma$ (spinors on T^d) is a set of spinors for each winding sector, labelled by the lattice Γ . These are the so called ‘tachyon states’, although, depending on the actual string theory at hand, they may not be tachyons (and hopefully they are not, as in superstring theory). The spaces \mathcal{F}^\pm are the Fock spaces of higher excitations (graviton, dilaton etc.) acted upon by the oscillator creation and annihilation modes. They represent the internal excitations of the strings and have an indirect effect on spacetime, which is described by the zero modes.

The description of interacting strings is done via the insertion on the world-sheet of *Vertex Operators*. The fundamental operator is the ‘tachyon vertex operator’

$$V_{q^\pm}(z_\pm) = : e^{-iq_i^\pm X_\pm^i(\tau \pm \sigma)} : \quad (16)$$

where $: \cdot :$ represents normal ordering obtained putting creators to the left of annihilators:

$$: \alpha_k^{(\pm)i} \alpha_m^{(\pm)j} : = \alpha_k^{(\pm)i} \alpha_m^{(\pm)j} \quad \text{for } k < m \quad (17)$$

$$= \alpha_m^{(\pm)j} \alpha_k^{(\pm)i} \quad \text{for } k > m \quad (18)$$

and x_{\pm}^i to the left of p_i^{\pm} . The tachyon vertex operator represents the insertion on the world sheet of a ground state (tachyon) of a given momentum. higher states (the dilaton, graviton etc.) are obtained acting with the appropriate combination of creation operators.

Vertex Operator Algebras have a distinguished place in mathematics, they have connections with modular functions, Monster Group, Lie Algebras and they are well reviewed in several publications, among wich [16]. I have no room to describe the beautiful mathematical intricacies of the theory, for most of our purposes in fact vertex operators will just be operators on \mathcal{H} .

One of the aspects of Vertex Operator Algebras which is important in this context is the *Operator–State correspondence*. We can put the generic vertex operator:

$$V(z_+, z_-)_{\psi} =: i V_{q^+ q^-}(z_+, z_-) \prod_j \frac{r_i^{(j)+}}{(n_j-1)!} \partial_{z_+}^{n_j} X_+^i \prod_k \frac{r_j^{(k)-}}{(m_k-1)!} \partial_{z_-}^{m_k} X_-^j : \quad (19)$$

in correspondence with the state:

$$|\psi\rangle = |q^+; q^-\rangle \otimes \prod_j r_i^{(j)+} \alpha_{-n_j}^{(+i)} |0\rangle_+ \otimes \prod_k r_j^{(k)-} \alpha_{-m_k}^{(-j)} |0\rangle_- \quad (20)$$

of \mathcal{H} , where $(q^+, q^-), (r^+, r^-) \in \Gamma \oplus \Gamma^*$.

We thus have the second element of the spectral triple, an algebra of vertex operator. A warning however: a vertex operator algebra (in the common use of the term) is *not a C^* algebra*. In general vertex operators are not even bounded operators! The problem stems from the fact that vertex operators are not defined at coinciding points giving rise to nontrivial *Operator Product Expansions* [2]. One can do two things to regularize the theory: *smear* the vertex operators [3, 17]:

$$V(\psi, f) = \int dz V_{\psi}(z) f(z) \quad (21)$$

but this not always cures the problem, as discussed for example in [18]. An alternative is to consider *truncated Vertex operators*:

$$V_{q^{\pm}}^N(z) = \mathcal{N}_N \prod_{n=0}^N W_n \quad (22)$$

where W_0 contains the zero modes x and p , while the W_n 's ($n \neq 0$) involve only the n^{th} oscillator modes $\alpha_n^{(\pm)}$ and $\alpha_{-n}^{(\pm)}$. This is equivalent to an ultraviolet cutoff on the world sheet, a standard practice in string theory to avoid

the infinities arising from the product of operators at coincident points. At the end one considers $N \rightarrow \infty$.

It is however fair to say that, at present, the rigorous definition of a C^* -algebra of operators representing interacting strings is (at least to my mind and my knowledge) still an open problem. We have the tachyon operators and the higher spin state (19), and one should regularize them, and create an algebra with the appropriate completion. It is in a sense like attempting to construct $C(\mathbb{R})$ from the knowledge of plane waves e^{ipx} . The general idea is present but many (crucial) details have to be filled. This is an area in which the collaboration of mathematicians would be of paramount importance. In the following we will indicate with the generic term vertex operator algebra a proper completion of the regularized operators.

We can easily recognize the two fundamental symmetries of the theory in the vertex operator algebra. As we said the tachyon operators are in a sense a “Fourier” or plane waves basis on the space of conformal field configurations. The tachyon states are highest weight states of the level a pair of $u(1)_+^d \oplus u(1)_-^d$ current algebra (14), so that the entire Hilbert space can be built up from the actions of the α_k ’s for $k < 0$ on these states. This current algebra represents the target space reparametrization symmetry of the string theory. On the other side, the two Virasoro algebras which represent the world sheet conformal invariance have irreducible representation whose highest weights grade the Hilbert space \mathcal{H} . The Virasoro operators in the present case are $L_k^\pm = \frac{1}{2} \sum_{m \in \mathbb{Z}} g_{ij} : \alpha_m^{(\pm)i} \alpha_{k-m}^{(\pm)j} :$, with $\alpha_0^{(\pm)i} \equiv g^{ij} p_j^\pm$. They generate a representation of the Virasoro algebra (13) of central charge d . The grading is defined on the subspaces $\mathcal{H}_{\Delta_q} \subset \mathcal{H}$ of states (20) which are highest weight vectors,

$$L_0^\pm |\psi\rangle = \Delta_q^\pm |\psi\rangle \quad , \quad L_k^\pm |\psi\rangle = 0 \quad \forall k > 0 \quad , \quad (23)$$

where $\Delta_q^+ = \frac{1}{2} g^{ij} q_i^+ q_j^+ + \sum_j n_j$ and $\Delta_q^- = \frac{1}{2} g^{ij} q_i^- q_j^- + \sum_k m_k$. The corresponding operator-valued distributions (19) are called primary fields.

The last element to complete the spectral triple is the Dirac operator. We have not one but *two* natural Dirac Operators:

$$D^\pm = \gamma_i^\pm \alpha_\pm^i \alpha_\pm^i = -i \partial_\pm X_\pm^i \quad (24)$$

These two operators generate target space reparametrization of X_\pm . Moreover it can be seen that they are square roots of the Laplace–Beltrami

operator. They are also naturally related to the other symmetry of the string theory, in fact worldsheet conformal symmetry has the conserved stress energy:

$$T^\pm(z_\pm) = -\frac{1}{2} : D^\pm(z_\pm)^2 : = \sum_k L_k^\pm z_\pm^{-k-2} \quad (25)$$

In analogy with the X and \tilde{X} we can define:

$$D = D^+ + D^- \quad ; \quad \tilde{D} = D^+ - D^- \quad (26)$$

The spectral triple \mathcal{T} of string geometry therefore is:

$$\mathcal{A} \quad \mathcal{H} \quad D \quad (27)$$

One can ask now what happened to ordinary spacetime? Spacetime emerges as a “subtriple” \mathcal{T}_0 , that is, a spectral triple with a subalgebra, a subspace of Hilbert space, and an operator which is the reduction of the Dirac operator on the subspace:

$$\mathcal{A}_0 \quad \mathcal{H}_0 \quad \tilde{D} \quad (28)$$

In order to construct the low energy subtriple we first have to project out all of the oscillator modes to obtain \mathcal{A}_0 and \mathcal{H}_0 . The rationale behind this is that, since the excited oscillators start at the Planck mass, and this is much larger than ‘ordinary’ space time energies, we have to isolate the modes of the string which will be accessible at low energies. This is still not sufficient however, as, in the case of large uncompactified directions, the modes associated to the winding are also highly energetic. We therefore choose:

$$\begin{array}{ll} C(\mathbb{R}^d) : & f \in \mathcal{A}_0 : \quad [\tilde{D}, f] = 0 \quad \text{commutant of } \tilde{D} \\ L_2(T^d, sp) : & \psi \in \mathcal{H}_0 : \quad \tilde{D}\psi = 0 \quad \text{kernel of } \tilde{D} \end{array} \quad (29)$$

It is easy to connect \mathcal{A}_0 with the algebra of complex valued function on spacetime, it is sufficient to notice that it is constructed from the (commutative) ‘vertex operator’ e^{ipx_0} . Here we encounter the already mentioned problems of the appropriate completion in order to obtain a well defined C^* -algebra . The essence of T-duality lies in the relatively simple observation the instead of \tilde{D} we could have chosen D as well. In this case we would have

obtained the triple pertaining to the torus whose coordinated are the \tilde{x} , that is the T-dual torus $\tilde{\mathcal{T}}_0$, with all the radii of compactifications inverted:

$$D^\pm \rightarrow D^\mp \quad D \leftrightarrow \tilde{D} \quad (30)$$

That the full theory is invariant under his change is ensured by the observation that this transformation is a *gauge* transformation. In fact there are (many) $u \in \mathcal{A}$ unitary such that:

$$uD u^{-1} = \tilde{D} \quad (31)$$

For example:

$$u = e^{i\mathcal{G}_\pm} \quad (32)$$

with

$$\mathcal{G}_\chi = \int \frac{dz_+ dz_-}{4\pi z_+ z_-} (\chi_{+,i}^a[X] J_a^{+(i)}(z_+) + \chi_{-,i}^a[X] J_a^{-(i)}(z_-)) f_S(z_+, z_-) \quad (33)$$

where $a = \pm$, $J_a^{\pm(i)} =: e^{aik_j^{(i)} X_\pm^j}$; χ_\pm are sections of the spin bundle and f_S is a smearing function.

This T-duality is a however a gauge transformation only in the full FG-triple. When this is projected to the subtriple so to give a *commutative* spacetime, in general the process will give rise to very different spacetimes. We can in fact consider T-duality to be the commutativity of the following diagram:

$$\begin{array}{ccc} \mathcal{T}_D & \xrightarrow{u} & \mathcal{T}_{\tilde{D}} \cong \mathcal{T}_D \\ \mathcal{P}_0 \downarrow & & \downarrow \tilde{\mathcal{P}}_0 \\ \mathcal{T}_0 & \xrightarrow{T_0} & \tilde{\mathcal{T}}_0 \end{array} \quad (34)$$

The operation T_0 is what we call T-Duality, and from the previous discussion it is clear that it is just the low energy projection of a gauge transformation. All of the remaining $O(d, d, \mathbb{Z})$ dualities can be obtained in the same way, as gauge transformations [5].

There are many more inner automorphisms, *gauge transformations*, which project down to non trivial transformations. Defining the currents:

$$J_\pm^i(\tau \pm \sigma) = \partial_\pm X_\pm^i(\tau \pm \sigma) = \sum_{k=-\infty}^{\infty} \alpha_k^{(\pm)i} e^{ik(\tau \pm \sigma)} \quad (35)$$

a general spacetime coordinate transformation $X \rightarrow \xi(X)$, with $\xi(X)$ a local section of $\text{spin}(T^n)$, is generated by $\mathcal{G}_\chi = \mathcal{G}_\xi$ with

$$\mathcal{G}_\xi = \int \frac{dz_+ dz_-}{4\pi z_+ z_-} \xi_i(X) (J_+^i(z_+) + J_-^i(z_-)) f_S(z_+, z_-) \quad (36)$$

This means that the also the diffeomorphisms of the (low energy) target space are gauge transformations of the full spectral triple. The inner automorphisms project down to outer automorphisms of spacetime. This is, in my opinion, one of the best justifications of the often heard statement that “general relativity is a gauge theory”. We can see also a glimpse of an huge group of symmetries, which when projected down connects different low energy theories.

If we try then to uncover the structure of spacetime at higher energies we would have to consider momentum and winding modes on a par. This will be relevant when the radius of compactification is comparable with Planck’s length, as in this case is not possible to ignore the former over the latter. We will however limit ourselves for the time being to the tachyonic case. Nevertheless the oscillators (at least the lower ones) do play an important role. Consider therefore tachyon vertex operators, for which we only excite the first N oscillators (for the basis e_\pm^i of $\gamma \oplus \Gamma^*$). The commutation relation among the elementary operators are:

$$\begin{aligned} V_{e_\pm^i}^N(z_{\pm i}) V_{e_\mp^j}^N(z_{\mp j}) &= V_{e_\mp^j}^N(z_{\mp j}) V_{e_\pm^i}^N(z_{\pm i}) \\ V_{e_\pm^i}^N(z_{\pm i}) V_{e_\pm^i}^N(z_{\pm i})^\dagger &= V_{e_\pm^i}^N(z_{\pm i})^\dagger V_{e_\pm^i}^N(z_{\pm i}) = \mathbb{I} \\ V_{e_\pm^i}^N(z_{\pm i}) V_{e_\pm^j}^N(z_{\pm j}) &= e^{2\pi i \omega_N^{ij}} V_{e_\pm^j}^N(z_{\pm j}) V_{e_\pm^i}^N(z_{\pm i}) \quad , \quad i \neq j \end{aligned} \quad (37)$$

where the z_\pm^i are distinct points, and

$$\omega_N^{ij} = \pm g^{ij} \left(\log \left(\frac{z_i^\pm}{z_j^\pm} \right) - \sum_{n=1}^N \frac{1}{n} \left(\left(\frac{z_i^\pm}{z_j^\pm} \right)^n - \left(\frac{z_j^\pm}{z_i^\pm} \right)^n \right) \right) . \quad (38)$$

One can easily recognize in (37) a *noncommutative torus* structure [19]. If we enclose more and more oscillators:

$$\lim_{N \rightarrow \infty} \omega_N^{ij} = \omega_\pm^{ij} = \pm g^{ij} \text{sgn}(\arg z_i^\pm - \arg z_j^\pm) \quad i \neq j \quad (39)$$

The symmetries of the theory are still present, even in this truncated version, in fact theories related by $O(d, d, \mathbb{Z})$ transformations give rise to Morita equivalent tori [7, 20]. The commutative case is recovered in the uncompactified/large compactification radius because when $R \rightarrow \infty$, the off diagonal elements of $g^{ij} \rightarrow 0$ and we recover the commutative torus.

“Turning the NCG crank” it is also possible to write a *Low Energy Dual Symmetric Action* [7]:

$$\begin{aligned} \mathcal{L} = & (F + \star F)_{ij} (F + \star F)^{ij} \\ & - i \bar{\psi} \gamma^i \left(\partial_i + i \overleftrightarrow{A}_i \right) \psi - i \bar{\psi} \tilde{\gamma}_i \left(\tilde{\partial}^i + i \overleftrightarrow{\tilde{A}}^i \right) \tilde{\psi} \end{aligned} \quad (40)$$

where the dual field strength is defined:

$$\begin{aligned} F_{ij} = & \partial_i A_j - \partial_j A_i + i [A_i, A_j] \\ & - g_{ik} g_{jl} \left(\tilde{\partial}^k \tilde{A}^l - \tilde{\partial}^l \tilde{A}^k + i [\tilde{A}^k, \tilde{A}^l] \right) \end{aligned} \quad (41)$$

All of the $O(d, d, \mathbb{Z})$ transformations, being unitary transformations, do not change the *spectrum* of D . Let us analyse in some details the transformation which changes the components of the antisymmetric second rank tensor b_{ij} by the addition of an arbitrary, integer valued, constant matrix. Although this transformation does not change the lattice Γ , it will change the momenta conjugated to the zero modes of X and \tilde{X} . In particular, in the spectrum (11), the relative contribution of the momenta (represented by the first term,) with respect to the windings, and the mixed term will change. Choosing the components of the antisymmetric tensor b arbitrarily large, we can make the contribution of the second and third term arbitrarily large. We have therefore concentrated the lowest eigenvalues of the Hamiltonian in the momentum part. The low energy spectrum is made only of the momentum eigenvalues. The lattice is still the same, but the strings are extremely twisted, and we have transferred the lowest eigenvalues of the energy from winding to momentum. Roughly speaking, a low energy strings, which in the original (small radius) lattice had a combination of momentum and winding, will now be twisted in such a way that it will appear to have just momentum, it is like the lattice “repeats itself over and over”.

Again, as in the case of the of the $R \leftrightarrow 1/R$ symmetry, we have to ask ourselves ‘what is position’? ‘How is it measured’? And using the same heuristic

arguments of [14], we can think of making wave packets using superpositions of the eigenvalues of the momentum. In the case of large torsion the eigenvalues of momentum are continuous for all practical purposes, therefore the superposition will have the character of a uncompactified space, rather than a string moving on a lattice. And this will be the situation until energies in which the new eigenvalues (coming from windings or the oscillatory modes) start to play a role.

Let us briefly discuss the role of the classical configuration space in ordinary quantum mechanics in the language and formalism of noncommutative geometry. We will be very brief and refer to [9] for further details and references. Consider a purely quantum observer, that is a set of operators which form an algebra. For example bounded operators constructed from p and x . The information on the topology of M , the manifold on which the motion is happening, can be recovered in the programme of noncommutative geometry by considering the algebra of *position operator*, that is, the algebra of continuous[‡], complex valued, functions on M , seen as operators on $L^2(M)$, with a norm given by the maximum of the modulus of the function. This is a simple application of the Gel'fand–Naimark theorem.

We will consider the configuration space of a quantum mechanical space therefore not as a set of points, but rather as an abelian C^* -algebra. The Hilbert space could also be easily constructed a posteriori by giving a sesquilinear form (a scalar product) on the algebra, and completing it under the norm given by this product. Other choice for the Hilbert space are possible, a relevant one for instance is the space of spinors. A quantum observer will have at his disposal, among the bounded operators on the Hilbert space, an abelian subalgebra \mathcal{A}_0 which he will identify with the continuous function on his space.

The “size” of this configuration space is given by the Dirac operator via Connes’ distance formula [1]:

$$d(x, y) = \sup_{\| [D, a] \| \leq 1} |a(x) - a(y)| \quad a \in \mathcal{A}_0 \quad . \quad (42)$$

Noncommutative geometry equips our quantum observer with a series of tools suited to him: algebras of operators, traces etc. In the commutative

[‡]In the following we will consider M compact, therefore continuous functions are bounded as well.

case these tools reconstruct the usual differential geometry, but they can be used in the noncommutative case as well. If we are in a commutative case, the quantum observer has therefore at his disposal an algebra of observables, in this algebra he recognizes an abelian subalgebra, that he calls the space on which he lives, and with formula (42) he calculates distances, metric etc.

In string theory, spacetime, as described for example by (37), the quantum observer finds himself on a *noncommutative space*. That is, among his set of quantum observable he does not identify an abelian algebra giving him the configuration space, he can however define some sort of “noncommutative” space, to which it corresponds a noncommutative algebra.

To specify the meaning of “low energy” we will resort to the spectral action principle [21], and will argue that the meaning of low energy means a theory in which only the low part of the spectrum of D plays a role. This is possible because in the framework of Noncommutative geometry one constructs a *spectral geometry*, in which the information is stored in the spectrum of D . And low energy refers to an action in which only the lower part of the spectrum is excited.

The spectral action principle is based on the covariant Dirac operator, and on the variation of its eigenvalues. The action must be read in a Wilson renormalization scheme sense, and it depends on an ultraviolet cutoff m_0 :

$$S_{m_0} = \text{Tr} \chi \left(\frac{D_A^2}{m_0^2} \right) \quad (43)$$

where D_A is the covariant Dirac operator and $\chi(x)$ is a function which is 1 for $x \leq 1$ and then goes rapidly to zero (some smoothened characteristic function). The action (43) effectively counts the eigenvalues of the covariant Dirac operator up to the cutoff. Considering, in fact, the eigenvalues of D_A as sequences of numbers, and these sequences as dynamical variables of euclidean gravity, the spectral action is then the action of “general relativity” in this space [22]. The trace in the action can be calculated using known heath kernel techniques [21], and the resulting theory contains a cosmological constant, the Einstein–Hilbert and Yang–Mills actions, plus some terms quadratic in the Riemann tensor.

What is important in the present context is the *spectral principle*, that is, the starting point is the spectrum of an operator, and its variations as the backgrounds fields (the one-form A in this case) change. One can ask,

in fact, what is the role of the algebra in the spectral action, as the latter depends just on the trace of the Dirac operator. Of course the role of the algebra is in the fact that in (43) appears the *covariant* Dirac operator. And the form $A = \sum a_i [D, b_i]$ depends on the algebra chosen. Let us now apply these considerations to the Fröhlich-Gawędzki spectral triple.

The spectrum of D and \tilde{D} , or of any operator obtained from them with an $O(d, d, \mathbb{Z})$ unitary transformation, are the same. Let us call D for convenience the one for which the lowest eigenvalues are the one relative to momentum. Here by lowest we mean the ones which are lower than the energy of the oscillatory modes (of the order of the Planck mass m_p). If the cutoff m_0 is lower than m_p , the cutoff function χ causes the projection of the operator on the Hilbert space \mathcal{H}_0 . Elements of the algebra which commutes with D (such as the elements of $\tilde{\mathcal{A}}$) will not contribute to the variations of the action, and will therefore be unobservable. This algebra can be constructed as the commutant of the T-dual operator \tilde{D} . This means that the winding modes degrees of freedom are unobservable. Since the Dirac operator has a near continuous spectrum, the tachyonic, low energy, algebra is spanned by operators of the kind

$$V_p = e^{ipx} \quad , \quad (44)$$

can be considered the Fourier modes describing an uncompactified space.

In fact, a quantum observer with a spectral action, will be able to measure (in the form of fields, potentials etc.) only the elements of the algebra which give low energy perturbations of the lowest eigenvalues of D , always with the assumption of the cutoff $m_0 < m_p$ so that oscillatory modes do not play a role. This is the abelian algebra of functions on some space time. If, as we have seen, there are many low eigenvalues, the observer will experience an effectively decompactified space time. The algebra which he will measure will be composed of the operators which will create low energy perturbation to D . At this point we have to make the sole assumption that D has a spectrum with several small eigenvalues. In this way the quantum observer will experience a (nearly) continuous spectrum of the momentum, the sign of an uncompactified space.

The strings could still be seen as compactified on a “small” lattice, but the presence of a very large torsion term b has drastically changed the operator content of the theory, and this has rendered space effectively uncompactified.

Conclusions

In both String Theory and Noncommutative Geometry, the interaction between physics and mathematics has been very fruitful, but the mathematics used in string theory has been essentially “classical” differential geometry. In this talk I tried to give an impressionistic way on how the mathematics well suited to describe strings in the high energy regime (which is proper to them) should be some sort of noncommutative geometry. While from the physical point of view some (initial) results are already to be seen: *duality, gauge transformations . . .*, from the mathematical point of view the structures to use are still in need of proper definitions.

A proper mathematical sharpening of the tools is necessary not so much for abstract mathematical rigour, but to help uncover the beauty which lies behind such a rich structure.

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